RADIATIVE HEAT EXCHANGE IN A CLOSED STATE CONSISTING OF THREE GRAY BODIES AND FILLED WITH A RADIATION-ABSORBING MEDIUM

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A solution is presented for the problem of radiative heat exchange between a gas and three gray surfaces in a closed space.

The literature [1, 2] cites the solution for the problem of radiative heat exchange in a closed system consisting of three gray surfaces. The system is filled with a radiation transparent (diathermic) medium.

Let us consider radiative heat exchange in a closed space consisting of three gray surfaces F_1, F_2, and F_3, when the angular coefficients $\varphi_{11} \neq 0$, $\varphi_{22} \neq 0$, and $\varphi_{33} \neq 0$. Within the space there is a radiation-absorbing medium (gas) exhibiting a uniform temperature throughout its entire volume. We regard the gas as gray, i.e., capable of absorbing radiant energy on all wavelengths, exhibiting no reflectance, but transparent to some extent for all wavelengths. We assume that the emissivity of the gas is different for the various radiation directions, and we denote it in the following manner:

$$\varepsilon_0^{12} = \varepsilon_0^{21},\ \varepsilon_0^{13} = \varepsilon_0^{31},\ \varepsilon_0^{23} = \varepsilon_0^{32},\ \varepsilon_0^{11},\ \varepsilon_0^{22},\ \varepsilon_0^{33}.$$

The subscript 0 for the ε denotes that the emissivity pertains to a gaseous medium, whereas the superscript shows the number of surfaces between which the radiation beam moves. We denote the gas temperature by T_0, while the temperature and emissivity of the surfaces F_1, F_2, and F_3, respectively, are denoted as T_1, T_2, T_3 and ε_1, ε_2, ε_3. We assume that $T_0 > T_1 > T_2 > T_3$.

The results of the exchange of heat betwen individual surfaces in the presence of a gas and with the participation of a third surface are expressed by the following equations [3]:

$$Q_{12} = F_1\varphi_{12}\varepsilon_{12}(E_1 - E_2) = F_2\varphi_{21}\varepsilon_{21}(E_1 - E_2), \quad (1)$$

$$Q_{13} = F_1\varphi_{13}\varepsilon_{13}(E_1 - E_3) = F_3\varphi_{31}\varepsilon_{31}(E_1 - E_3), \quad (2)$$

$$Q_{23} = F_2\varphi_{23}\varepsilon_{23}(E_2 - E_3) = F_3\varphi_{32}\varepsilon_{32}(E_2 - E_3). \quad (3)$$

The results of the exchange of heat between the gas and each of the surfaces, with the participation of two other surfaces, are represented by the functions [3]:

$$Q_{01} = F_1\varepsilon_{01}(E_0 - E_1), \quad (4)$$

$$Q_{02} = F_2\varepsilon_{02}(E_0 - E_2), \quad (5)$$

$$Q_{03} = F_3\varepsilon_{03}(E_0 - E_3), \quad (6)$$

where

$$E_n = 4.96\left(\frac{T_n}{100}\right)^4 \quad (n = 0,\ 1,\ 2,\ 3),$$

E_n is the specific emissive power of an absolute black body at the temperature T_n; n corresponds to the notation of the gas or surface; $\varepsilon_{12} = \varepsilon_{21}$, $\varepsilon_{13} = \varepsilon_{31}$, $\varepsilon_{23} = \varepsilon_{32}$ are the general emissivities for the particular systems consisting of two surfaces. The subscript with the ε indicates the number of surfaces making up the system; ε_{0n} (n = 1, 2, 3) is the general emissivity of the gas-surface system.

To solve the formulated problem, we must determine the emissivities of the particular systems ε_{12}, ε_{13}, ε_{23}, ε_{01}, ε_{02}, and ε_{03}.

The solution of the problems of radiative heat exchange can be achieved algebraically if in a system of p surfaces the following condition is satisfied [4–6]:

$$\varphi_{dn-k} = \varphi_{n-k} \quad (n,\ k = 1,\ 2,\ 3,\ \ldots,\ p), \quad (7)$$

which indicates that the angular coefficient from any unit area of surfaces F_n to F_k is equal to the mean angular coefficient from F_n to F_k. As an example of systems for which (7) is satisfied we cite a sphere within a sphere, with a common center, or two infinitely long cylinders with a common axis. If (7) is not satisfied, we are forced to deal with integral equations. We assume that (7) is satisfied for our system.

We use the method described in [7] to solve the problem; according to this method we derive the energy balance for each of the three surfaces, assuming conditionally for the moment that the surface F_1 has a temperature $T_1 \neq 0$, while the temperature of the remaining two surfaces and that of the gas are equal to absolute zero. As a result we have an algebraic system of three equations with three unknowns:

$$\left.\begin{aligned}
&\left(H_{11}d_0^{11} - \frac{F_1}{r_1}\right){}_1R_1 + H_{12}d_0^{12}\,{}_1R_2 + H_{13}d_0^{13}\,{}_1R_3 = \\
&\qquad = -H_{11}d_0^{11}\,\varepsilon_1, \\
&H_{12}d_0^{12}\,{}_1R_1 + \left(H_{22}d_0^{22} - \frac{F_2}{r_2}\right){}_1R_2 + H_{23}d_0^{23}\,{}_1R_3 = \\
&\qquad = -H_{12}d_0^{12}\,\varepsilon_1, \\
&H_{13}d_0^{13}\,{}_1R_1 + H_{23}d_0^{23}\,{}_1R_2 + \left(H_{33}d_0^{33} - \frac{F_3}{r_3}\right){}_1R_3 = \\
&\qquad = -H_{13}d_0^{13}\,\varepsilon_1,
\end{aligned}\right\} \quad (8)$$

where

$${}_1R_n = \frac{{}_1E_{n\,\mathrm{eff}}}{E_1} \quad (n = 1,\ 2,\ 3); \quad (9)$$

n is the surface number; ${}_1E_{neff}$ is the density of the total (effective) heat flux from the surface F_n, i.e., the intrinsic and the reflected flow; ${}_1R_n$ is the relative density of the total heat flux from the surface F_n. The left superscript indicates the original radiation source, while the right subscript denotes the surface from which the effective heat flux emanates; H denotes the equivalent (mutual) surface in the exchange of heat between two surfaces: $H_{11} = F_1\varphi_{11}$; $H_{12} = F_1\varphi_{12} = F_2\varphi_{21}$ etc.; $d_0^{11} = 1 - \varepsilon_0^{11}$, $d_0^{12} = 1 - \varepsilon_0^{12}$, etc., denotes the transparency of the gas on motion of the beam in various directions according to the superscript; $r_1 = 1 - \varepsilon_1$, $r_2 = 1 - \varepsilon_2$, and $r_3 = 1 - \varepsilon_3$ are the reflectances of the surfaces F_1, F_2, and F_3.

The emissivity ε_{12} is determined from the equation

$$F_1\varphi_{12}\varepsilon_{12} = F_2\frac{\varepsilon_2}{r_2}\,{}_1R_2. \tag{10}$$

In the left-hand part of (10) we find the result of the heat exchange, measured in units of E_1, between F_1 and F_2, derived from (1) when $T_2 = 0$. In the right-hand part the quantity $F_2 = {}_1R_2/r_2$ represents the relative heat flux incident on the surface F_2, since the magnitude of the relative density ${}_1R_2$ of the total heat flux from F_2 when $T_2 = T_3 = T_0 = 0$ is equal to the relative density of the reflected heat flux from F_2 alone. The product of $F_2({}_1R_2/r_2)$ by ε_2 shows the relative heat flux absorbed by the surface F_2 as a result of heat exchange.

From (10) for ε_{12} we obtain

$$\varepsilon_{12} = \frac{F_2\varepsilon_2}{F_1\varphi_{12}r_2}\,{}_1R_2. \tag{11}$$

The quantity ε_{12} determined from (11) according to the adopted working model in which $T_0 = T_2 = T_3 = 0$ will also be valid for the case in which the temperatures T_0, T_2, and T_3 are different from zero, since ε_{12} is a function of the physical and geometric constants and independent of temperature. Equations analogous to (11) are also derived for the determination of ε_{13} and ε_{23}:

$$\varepsilon_{13} = \frac{F_3\varepsilon_3}{F_1\varphi_{13}r_3}\,{}_1R_3, \tag{12}$$

$$\varepsilon_{23} = \frac{F_3\varepsilon_3}{F_2\varphi_{23}r_3}\,{}_2R_3, \tag{13}$$

where ${}_2R_3 = {}_2E_{3eff}/E_2$ is the relative density of the total heat flux from F_3 under the condition that the surface F_2 exhibits a temperature $T_2 \neq 0$, and the temperatures $T_1 = T_3 = T_0 = 0$.

To determine ${}_2R_3$ we compile a system of equations analogous to (8) from which the quantity ${}_2R_3$ is determined. The expressions for ${}_1K_2$ and ${}_1R_3$ are determined from (8).

Having substituted the expressions for ${}_1R_2$, ${}_1R_3$, and ${}_2R_3$, respectively, into (11), (12), and (13), we obtain the final expressions for the determination of ε_{12}, ε_{13}, and ε_{23}:

$$\varepsilon_{nk} = \Big(\varepsilon_n\varepsilon_k\,[\varphi_{nk}\,d_0^{nk}\,(\varphi_{mm}\,d_0^{mm}\,r_m - 1) - \varphi_{nm}\varphi_{mk}\,d_0^{nm}\,d_0^{mk}\,r_m]\Big) \times \Big(A\Big)^{-1}, \tag{14}$$

where nk represents a combination of the three elements 1, 2, and 3, denoting the surface numbers, two each, while m is the number of the remaining surface not included in the combination nk;

$$A = (\varphi_{11}d_0^{11}r_1 - 1)(\varphi_{22}d_0^{22}r_2 - 1)(\varphi_{33}d_0^{33}r_3 - 1) + 2\varphi_{12}\varphi_{23}\varphi_{31}d_0^{12}d_0^{13}d_0^{23}r_1r_2r_3 - \varphi_{23}\varphi_{32}(d_0^{23})^2 \times (\varphi_{11}d_0^{11}r_1 - 1)r_2r_3 - \varphi_{13}\varphi_{31}(d_0^{13})^2(\varphi_{22}d_0^{22}r_2 - 1)r_1r_3 - \varphi_{12}\varphi_{21}(d_0^{12})^2(\varphi_{33}d_0^{33}r_3 - 1)r_1r_2. \tag{15}$$

According to [7], to determine the emissivities ε_{01}, ε_{02}, and ε_{03} we can make up the following system of equations based on the energy balance for each of the surfaces, if we assume that the gas has a temperature $T_0 \neq 0$, with the surface temperatures equal to zero ($T_1 = T_2 = T_3 = 0$):

$$\left.\begin{aligned}
&\left(H_{11}d_0^{11} - \frac{F_1}{r_1}\right){}_0R_1 + H_{12}d_0^{12}\,{}_0R_2 + H_{13}d_0^{13}\,{}_0R_3 = -\varepsilon_0^{equ1}F_1,\\
&H_{12}d_0^{12}\,{}_0R_1 + \left(H_{22}d_0^{22} - \frac{F_2}{r_2}\right){}_0R_2 + H_{23}d_0^{23}\,{}_0R_3 = -\varepsilon_0^{equ2}F_2,\\
&H_{13}d_0^{13}\,{}_0R_1 + H_{23}d_0^{23}\,{}_0R_2 + \left(H_{33}d_0^{33} - \frac{F_3}{r_3}\right){}_0R_3 = -\varepsilon_0^{equ3}F_3,
\end{aligned}\right\} \tag{16}$$

where ${}_0R_n = {}_0E_{neff}/E_0$ (n = 1, 2, 3) is the relative density of the total (effective) heat flux from the surfaces of the system in which the sole source of radiation is a radiation-absorbing medium having the temperature $T_0 \neq 0$, and $T_1 = T_2 = T_3 = 0$; ε_0^{equ1}, ε_0^{equ2}, ε_0^{equ3} are determined from the equations

$$\varepsilon_0^{equ\,n} = \varphi_{n1}\varepsilon_0^{n1} + \varphi_{n2}\varepsilon_0^{n2} + \varphi_{n3}\varepsilon_0^{n3} \quad (n = 1,\ 2,\ 3), \tag{17}$$

$\varepsilon_0^{equ\,n}$ is the equivalent emissivity of the gas for the radiation proceeding from the surface F_n in all three directions.

The equations for the determination of the emissivities ε_{01}, ε_{02}, and ε_{03} are found in analogy with (11), (12), and (13):

$$\varepsilon_{0n} = \frac{\varepsilon_n}{r_n}\,{}_0R_n \quad (n = 1,\ 2,\ 3), \tag{18}$$

where n is the surface number.

Having substituted into (18) the expressions for ${}_0R_1$, ${}_0R_2$, and ${}_0R_3$, determined from (16), we derive a generalized expression for the determination of ε_{01}, ε_{02}, and ε_{03}:

$$\varepsilon_{0n} = \frac{\varepsilon_n}{A}\,\{[\varphi_{mk}\varphi_{km}(d_0^{mk})^2 r_m r_k -$$

$$-(\varphi_{mm} d_0^{mm} r_m - 1)(\varphi_{kk} d_0^{kk} r_k - 1)\Big]\, \varepsilon_0^{\text{equ}\,n} +$$
$$+ \Big[\varphi_{nm} d_0^{nm} r_m (\varphi_{kk} d_0^{kk} r_k - 1) -$$
$$- \varphi_{nk}\varphi_{km} d_0^{nk} d_0^{km} r_m r_k\Big]\, \varepsilon_0^{\text{equ}\,m} +$$
$$+ \Big[\varphi_{nk} d_0^{nk} r_k (\varphi_{mm} d_0^{mm} r_m - 1) -$$
$$- \varphi_{nm}\varphi_{mk} d_0^{nm} d_0^{mk} r_m r_k\Big]\, \varepsilon_0^{\text{equ}\,k}\Big\}. \quad (19)$$

Here n = 1, 2, 3 is the number of the surface forming—with the gas—a particular system for which we have to determine the emissivity; m and k are the numbers of the two remaining surfaces not included in the gas-surface system. For example, if n = 1, m = 2 and k = = 3; if n = 2, m = 1, k = 3; if n = 3, m = 1, k = 2. As before, A is calculated from (15).

Having substituted the derived expressions for ε_{12}, ε_{13}, ε_{23}, ε_{01}, ε_{02}, and ε_{03} from (14) and (19), respectively, into (1), (2), (3) and (4), (5), (6), we can calculate the results of the radiative heat exchange in a closed space consisting of three gray bodies and filled with a radiation-absorbing medium. The resulting solutions of (14) and (19) for the emissivities of the systems are the most general. The special cases can be derived from them. For example, when the emissivity of the gas is identical in all direction, i.e., $\varepsilon_0^{11} = \varepsilon_0^{22} = \varepsilon_0^{33} = \varepsilon_0^{12} = \varepsilon_0^{13} = \varepsilon_0^{23} = \varepsilon_0$, the solutions of (14) and (19) assume the form

$$\varepsilon_{nk} = \frac{\varepsilon_n \varepsilon_k [\varphi_{nk} d_0 (\varphi_{mm} d_0 r_m - 1) - \varphi_{nm}\varphi_{mk} d_0^2 r_m]}{B}, \quad (20)$$

where nk, as before, represents a combination of three elements 1, 2, and 3, two each, while m is the number of the surface not included in the combination nk;

$$\varepsilon_{0n} = \frac{\varepsilon_0 \varepsilon_n}{B}\Big[\varphi_{mk}\varphi_{km} d_0^2 r_m r_k -$$
$$-(\varphi_{mm} d_0 r_m - 1)(\varphi_{kk} d_0 r_k - 1) +$$
$$+ \varphi_{nm} d_0 r_m (\varphi_{kk} d_0 r_k - 1) - \varphi_{nk}\varphi_{km} d_0^2 r_m r_k +$$
$$+ \varphi_{nk} d_0 r_k (\varphi_{mm} d_0 r_m - 1) - \varphi_{nm}\varphi_{mk} d_0^2 r_m r_k\Big],$$

$$\varepsilon_0^{\text{equ}\,n} = (\varphi_{n1} + \varphi_{n2} + \varphi_{n3})\varepsilon_0 = \varepsilon_0, \quad (21)$$

since $\varphi_{n1} + \varphi_{n2} + \varphi_{n3} = 1$, where n, m, and k are the same as in the solution (19):

$$d_0 = 1 - \varepsilon_0;$$
$$B = (\varphi_{11} d_0 r_1 - 1)(\varphi_{22} d_0 r_2 - 1)(\varphi_{33} d_0 r_3 - 1) +$$
$$+ 2\varphi_{12}\varphi_{23}\varphi_{31} d_0^3 r_1 r_2 r_3 - \varphi_{23}\varphi_{32} d_0^2 (\varphi_{11} d_0 r_1 - 1) r_2 r_3 -$$
$$- \varphi_{13}\varphi_{31} d_0^2 (\varphi_{22} d_0 r_2 - 1) r_1 r_3 -$$
$$- \varphi_{12}\varphi_{21} d_0^2 (\varphi_{33} d_0 r_3 - 1) r_1 r_2. \quad (22)$$

If we assume that the medium in the system is transparent to heat (for example, dry air), $\varepsilon_0^{11} = \varepsilon_0^{22} = \varepsilon_0^{33} = \varepsilon_0^{12} = \varepsilon_0^{13} = \varepsilon_0^{23} = \varepsilon_0 = 0$ and the resulting solutions, by means of algebraic transformations, are changed into the solution known from the literature [1] for a system of three surfaces, with the system filled with a radiation-transparent medium.

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